The structure of n-variable polynomial rings as Hecke algebra modules

1993 J. Phys. A: Math. Gen. 26 7311

(http://iopscience.iop.org/0305-4470/26/24/009)

J. Phys. A: Math. Gen. 26 (1993) 7311–7324. Printed in the UK

The structure of n-variable polynomial rings as Hecke algebra modules

Paul Martin

Mathematics Department, City University, Northampton Square, London EC1V 0HB, UK

Received 6 January 1993, in final form 9 August 1993

Abstract. We determine the structure of the ring of n-variable polynomials as a module for the Hecke algebra $H_n(q)$ (q an invertible parameter). We show how to construct polynomial bases for the q analogues of symmetric group Specht modules, fully generalizing the relevant result of Davies *et al* (on two representations in the Temperley–Lieb quotient).

1. Introduction

This paper is motivated by the need, illustrated particularly in section 6 of the important recent work of Davies *et al* [1] on the spectrum of the XXZ Hamiltonian, and by recent interest in Hecke algebras and q-Knizhnik–Zamolodchikov equations from various authors [2], for a brief 'physicist digestible' treatment of the structure of the ring of polynomials in n variables as a Hecke algebra module.

Recall that if R is a ring then $R[x]$ is the ring of polynomials in x with coefficients in R, and $R[x, y] = R[x][y]$. Now let $\mathbb{Z}_q = \mathbb{Z}[q, q^{-1}]$ and let $\mathbb{P} = \mathbb{Z}_q[z_1, z_2, \ldots, z_n]$ be the associated ring of polynomials in n independent variables. The generators $\{g_i;\ i = 1, 2, \ldots, n-1\}$ of the unitary Hecke algebra $H_n(q)$ over $\mathbb{Z}_q$ with relations

$$\begin{aligned} &g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \\ &[g_i, g_j] = 0 \qquad |i-j| \neq 1 \\ &(g_i + q)(g_i - q^{-1}) = 0 \end{aligned} \tag{1}$$

act on the ring $\mathbb{P}$ by a q-deformation of the action of the symmetric group $S_n \cong H_n(\pm 1)$ on the variables $z_1, z_2, \ldots, z_n$. For f a function of $z_1, z_2, \ldots, z_n$ and $(i\ \ i+1) \in S_n$ let $f^{(i)}$ denote

$$\begin{aligned} f^{(i)}(z_1, z_2, \ldots, z_i, z_{i+1}, \ldots, z_n) &= (i\ \ i+1) f(z_1, z_2, \ldots, z_i, z_{i+1}, \ldots, z_n) \\ &= f(z_1, z_2, \ldots, z_{i+1}, z_i, \ldots, z_n). \end{aligned}$$

Then explicit computation shows that

$$(g_i + q) f = (q^{-1} z_i - q z_{i+1}) \frac{(f - f^{(i)})}{(z_i - z_{i+1})} \tag{2}$$

defines an action of $H_n(q)$ from the left on the set of such functions [3, 1]. If $f \in \mathbb{P}$ then $f - f^{(i)}$ is a polynomial with a zero at $z_i = z_{i+1}$, so $(f - f^{(i)})/(z_i - z_{i+1}) \in \mathbb{P}$ and $\mathbb{P}$ is an invariant subspace as an $H_n(q)$ module.

In this paper $\mathbb{N}$ denotes the set of non-negative integers. Let $a = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ and define $|a| = \sum_k a_k$. Then a basis of $\mathbb{P}$ is the monomials

$$\mathcal{M} = \{z^a = z_1^{a_1} \ldots z_n^{a_n} : a \in \mathbb{N}^n\}.$$

For any $i \in \{1, 2, \ldots, n-1\}$ and $a \in \mathbb{N}^n$ let $\alpha \in \mathbb{N}^n$ be given by $\alpha_j = a_j$ $(j \neq i, i+1)$, $\alpha_i = \alpha_{i+1} = \min\{a_i, a_{i+1}\}$. Then from equation (2)

$$(g_i + q)z^a = (q^{-1}z_i - qz_{i+1})z^\alpha \frac{(\mathrm{sign}(a_i - a_{i+1})(z_{i+1}^{|a_{i+1}-a_i|} - z_i^{|a_{i+1}-a_i|}))}{(z_i - z_{i+1})}$$

so factorizing

$$\begin{aligned}(g_i + q)z^a &= (q^{-1}z_i - qz_{i+1})z^\alpha \left(\mathrm{sign}(a_i - a_{i+1}) \left(\sum_{k=1}^{|a_{i+1}-a_i|} z_i^{|a_{i+1}-a_i|-k} z_{i+1}^{k-1} \right) \right) \\ &= \mathrm{sign}(a_i - a_{i+1}) z^\alpha \left(q^{-1} z_i^{|a_{i+1}-a_i|} + \left((q^{-1} - q) \sum_{k=1}^{|a_{i+1}-a_i|-1} z_i^{|a_{i+1}-a_i|-k} z_{i+1}^k \right) \right. \\ &\quad \left. - q z_{i+1}^{|a_{i+1}-a_i|} \right). \end{aligned} \tag{3}$$

It follows, since $|a|$ is the total degree of every monomial in the expansion of the right-hand side of equation (3), that as an $H_n(q)$ module

$$\mathbb{P} = \bigoplus_{k \in \mathbb{N}} \mathbb{P}^k$$

where $\mathbb{P}^k$ is the space of polynomials of order k (i.e. with elements homogeneous of degree k).

Indeed for each $j = 1, 2, \ldots, n$ the power of z_j in each term on the right-hand side of equation (3) is no greater than a_j, so within $\mathbb{P}^k$ the monomials z^a with all $a_j < Q$ $(Q \in \mathbb{N})$ span an invariant subspace, call it $\mathbb{P}^k(Q)$. Then we have inclusions of $H_n(q)$ modules

$$\mathbb{P}^k(Q) \subset \mathbb{P}^k(Q+1).$$

In section 2 we analyse the structure of these modules. Many recent papers point to the importance of Hecke algebra representations in exactly solvable models, knot theory and conformal field theory [2, 4]. It is well known how to construct irreducible representations of $H_n(q)$ for q not a root of unity, but the usual constructions (first obtained by generalizing Young's classical S_n work by Hoefsmit [5], implicit in Andrews *et al* [6] and subsequently rediscovered by several others [7]) are not well defined at q a root of unity. Indeed this can be regarded as a signal of the change in the structure of the algebra at these points [8]. The more robust constructions of James and others [8, 9] solve this problem, but are very difficult to implement in practice. One of the benefits of the present approach is that by bringing in the idea of symmetric functions we can put the procedure into a manageable state. This is done in the final section.

2. Physical contexts

In [1] Davies *et al* give a scheme to diagonalize the one-dimensional XXZ spin chain Hamiltonian

$$\mathcal{H}_{\mathrm{XXZ}} = -\tfrac{1}{2} \sum_{k=-\infty}^{\infty} (\sigma_{k+1}^x \sigma_k^x + \sigma_{k+1}^y \sigma_k^y + \Delta \sigma_{k+1}^z \sigma_k^z) \tag{4}$$

for the (anti-ferroelectric) region $\Delta < -1$, using the representation theory of the quantum affine algebra $U_q(\widehat{sl}(2))$, where $\Delta = (q + q^{-1})/2$. On a finite lattice with appropriate boundary conditions the XXZ model can be built from reducible representations of Hecke algebras. In such cases $\mathcal{H}$ commutes with an action of the ordinary quantum enveloping algebra $U_q(sl(2))$ [13]. In the thermodynamic limit, as in equation (4), this Hamiltonian formally acts on the infinite tensor product

$$W = \ldots \otimes C^2 \otimes C^2 \otimes C^2 \otimes C^2 \otimes C^2 \otimes \ldots$$

(in the sense that the Pauli matrix σ_k^x acts on the kth space C^2 here) and commutes with the action of $U_q'(\widehat{sl}(2)) \subset U_q(\widehat{sl}(2))$ on this space† (see [1] for details, and see also the continuum limit version—the $su(2)$-invariant Thirring model [10]). Davies *et al* [1] use this commuting property to diagonalize the Hamiltonian (the technical issues are extensive, and for complete details the reader should again refer to their paper—here we touch only on the aspects germane to the present paper). In order to make their analysis rigorous Davies *et al* [1] initiate a study of the n point correlators of vertex operators (certain intertwiners of representations of $U_q(\widehat{sl}(2))$) as n grows large. Frenkel and Reshetikhin [11] showed that these correlators satisfy a q-Knizhnik–Zamolodchikov equation. It follows indirectly that these correlators provide bases for representations of a Hecke algebra, and in special cases (small n) Davies *et al* [1] observe that these are polynomial bases generalizing some well known symmetric group bases. They leave their result for large n as a conjecture and, in any case, the appropriate generalizations were not previously known. They are given in this paper in a self-contained presentation.

This is of interest not only for the spectrum of the XXZ model itself (and its interpretation) but also because of the scope for using the technique in more general models. There are technical problems in each of these aspects. For a full explanation the reader should turn to the original paper, but we may summarize them as (i) it is not known how to generalize from $sl(2)$ to $sl(N)$; and (ii) there is ambiguity in the overall normalization of vertex operators, leading to a convergence issue for their infinitely iterated application (as in the thermodynamic limit). The results given in this paper provide the framework for addressing both of these issues, as well as that of eigenvectors for XXZ and higher vertex models in other regions. This framework can also be used to study eigenvectors of a suitable formulation of the Calogero–Sutherland–Moser model [12], which may be relevant to the quantum Hall effect and high T_c superconductivity, which we will discuss elsewhere.

We will show shortly that many physical models (arbitrary $U_q(sl(N))$ spin chains, for example) may be expressed in the form of the n site chain Hamiltonian

$$\mathcal{H}_\nabla(\alpha_1, \alpha_2, \ldots, \alpha_{n-1}) = \sum_{k=1}^{n-1} \alpha_k (q^{-1} z_k - q z_{k+1}) \nabla^{(k)} \tag{5}$$

† Explicitly the action is defined in two stages: first a representation $\pi : U_q'(\widehat{sl}(2)) \to \mathrm{End}(C^2)$ is given by representing the generators:

$$\pi(e_0) = \pi(f_1) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \pi(e_1) = \pi(f_0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \pi(t_1) = \pi(t_0^{-1}) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

and then the action of $U_q'(\widehat{sl}(2))$ on W is formally determined by the 'infinite' comultiplication

$$\Delta^{(\infty)}(e_i) = \sum_{k \in Z} \ldots \otimes t_i \otimes e_i \otimes 1 \ldots \qquad \Delta^{(\infty)}(f_i) = \sum_{k \in Z} \ldots \otimes 1 \otimes f_i \otimes t_i^{-1} \ldots$$

$$\Delta^{(\infty)}(t_i) = \ldots \otimes t_i \otimes t_i \otimes t_i \ldots .$$

where the operator $\nabla^{(i)} = [1-(i\ i+1)]/(z_i - z_{i+1})$, the αs are coupling constants, and the Hamiltonian acts on some subspace of the formal space of functions of n variables z_i. We could attempt to consider the solutions to the general homogeneous problem

$$\mathcal{H}f = Ef$$

where $\mathcal{H} = \mathcal{H}_\nabla(\alpha, \alpha, \dots, \alpha)$, but this is not, as it turns out, physically sensible. The different models mentioned correspond to restricting to quotients of different invariant subspaces of the space of functions, and these subquotients yield different parts of the eigenvalue spectrum. In the thermodynamic limit no single model sees the whole spectrum, thus it is not sensible to try to give the complete spectrum of $\mathcal{H}$ a unified physical interpretation. This is an illuminating paradigm for the Calogero–Sutherland–Moser model [12] which is much harder to solve but which has, in principle, a similarly nebulous function space (essentially the same one, in fact, see [12]).

In this paper we will, however, fortuitously obtain several examples of eigenfunctions of $\mathcal{H}$ (and even of $\mathcal{H}_\nabla(\alpha_1, \alpha_2, \dots, \alpha_{n-1})$)—the first, and simplest examples occur in equation (18), for instance. These are already known via the Bethe ansatz [14], but our representation theoretic derivation offers the possibility of an independent approach.

3. The structure of $\mathbb{P}^k$

We begin with some more notation: recall that a partition of a set A is any collection of disjoint subsets A_i whose union is A, and that each equivalence relation on A defines such a partition.

On the other hand, for $n \in \mathbb{N}$, a partition of n is any list $\lambda = (\lambda_1, \lambda_2, \dots)$ with each $\lambda_i \in \mathbb{N}$ (usually $\lambda_i > 0$), $\sum_i \lambda_i = n$ and $\lambda_i \geqslant \lambda_{i+1}$ [15]. If λ is a partition of n we write $\lambda \vdash n$.

The *depth* of $\lambda \vdash n$ is the unique index i such that $\lambda_i > 0$ but $\lambda_{i+1} \not> 0$. The set of partitions of n is $\mathcal{P}_n$.

For $\lambda \vdash n$ the conjugate $\lambda' \vdash n$ is defined by

$$\lambda'_j = \mathrm{card}\{i : \lambda_i \geqslant j\}.$$

Recall also the reverse lexicographic ordering of partitions of n given by $\lambda \rhd \mu$ if the first non-vanishing difference $\lambda_i - \mu_i$ is positive [15]. This is a total ordering of partitions of n.

Now for each $a \in \mathbb{N}^n$ let us define $\rho(a)$ as the partition of $A = \{1, 2, \dots, n\}$ given by the equivalence relation

$$i \sim_a j \qquad \text{iff } a_i = a_j.$$

Then:

Definition 1 (Profile). For $a \in \mathbb{N}^n$ define the 'profile' of a, written ρ^a, as the partition of n given by the list of orders of subsets in the partition $\rho(a)$, i.e. $\rho^a = (|A_1|, |A_2|, ..)$, with subsets arranged so that $|A_i| \geqslant |A_{i+1}|$.

Definition 2 (Shape). The 'shape' of an $a \in \mathbb{N}^n$, written $a^\square$, is the partition of $|a|$ obtained by permuting the elements of a (i.e. rearranging the $\{a_i\}$ until $a_1 \geqslant a_2 \geqslant \dots$).

We also say that z^a has shape $a^\square$. Note that for given n the depth of this partition $a^\square$ cannot exceed n.

For example, with $n = 5$, consider $a = (4,5,1,0,0)$. Then $z^a = z_1^4 z_2^5 z_3$, the shape of a is $(5,4,1,0,0)$, $|a| = 10$, and $\rho^a = (2,1,1,1)$. If $b = (4,4,2,0,0)$ then the shape is $(4,4,2,0,0)$, $|b| = 10$ and $\rho^b = (2,2,1)$; $c = (6,2,7,0,0)$ gives $c^\square = (7,6,2,0,0)$, $|c| = 15$, $\rho^c = (2,1,1,1)$.

Proposition 1. For any monomial z^a the shape $a^\square$ is greater or equal in the reverse lexicographic ordering to the shape of any monomial in the expansion of $g_i\ z^a$ (from the right-hand side of equation (3)).

Proof. For each i there are three cases to consider:

For $a_i = a_{i+1}$ then $f - f^{(i)} = 0$.

For $a_i > a_{i+1}$ then a_i precedes a_{i+1} in $a^\square$. The first basis state occurring on the right-hand side (apart from z^a itself), call it z^b, has a partition of the form

$$b^\square = (\ldots, a_i - 1, \ldots, a_{i+1} + 1, \ldots)$$

(as a partition the ith and $(i+1)$th components of b may no longer be adjacent). Thus $a^\square \rhd b^\square$. All subsequent terms are contained in $g_i z^b$, and hence have shapes below $a^\square$ by the same argument, except $(\ldots, a_{i+1}, \ldots, a_i, \ldots)$ which is identical to a up to permutation.

For $a_i < a_{i+1}$ similarly. $\square$

It follows that $\mathbb{P}^i$ has a sequence of invariant subspaces as a left $H_n(q)$ module. For $\lambda \vdash i$ let $\mathbb{P}^\lambda$ be the space of polynomials of shape λ (for $n = 3, i = 3$ the space $\mathbb{P}^{(2,1,0)}$ is spanned by

$$\{z_1^2 z_2, z_1^2 z_3, z_1 z_2^2, z_1 z_3^2, z_2^2 z_3, z_2 z_3^2\}$$

for example), and let

$$\mathbb{P}_+^\lambda = \mathbb{P}^\lambda + \sum_{\{\mu:\mu_1' \leqslant n \quad \mu \lhd \lambda\}} \mathbb{P}^\mu. \tag{6}$$

For example (with $i \leqslant n$)

$$\mathbb{P}_+^{(1^i)} = \mathbb{P}^{(1^i)} \qquad \mathbb{P}_+^{(21^{i-2})} = \mathbb{P}^{(21^{i-2})} + \mathbb{P}^{(1^i)}.$$

Then:

Proposition 2. For $\lambda, \mu \vdash i$ and $\lambda \rhd \mu$

$$\mathbb{P}_+^\mu \subset \mathbb{P}_+^\lambda \subseteq \mathbb{P}^i$$

are inclusions of left $H_n(q)$ modules.

For $i \leqslant n$ we have left $H_n(q)$ module inclusions

$$\mathbb{P}_+^{(1^i)} \subset \mathbb{P}_+^{(21^{i-2})} \subset \mathbb{P}_+^{(2^2 1^{i-4})} \subset \ldots \subset \mathbb{P}_+^\mu \subset \mathbb{P}_+^{\lambda(\rhd\mu)} \subset \ldots \subset \mathbb{P}^i.$$

For $i > n$ with $i = nm + k$ $(m, k \in \mathbb{N})$ then the smallest invariant subspace in this filtration is $\mathbb{P}^\omega$ with

$$\omega = ((m+1)^k, m^{n-k}).$$

Let

$$\mathbb{P}^{\lambda}_{-} = \mathbb{P}^{\lambda}_{+} \Big/ \Big(\sum_{\mu \lhd \lambda} \mathbb{P}^{\mu} \Big). \tag{7}$$

Then $\mathbb{P}^{\lambda}_{-}$ is an $H_n(q)$ module, while $\mathbb{P}^{\lambda}$, which can have the same basis (and is isomorphic to $\mathbb{P}^{\lambda}_{-}$ as a vector space), is not an $H_n(q)$ module in general. For example $\mathbb{P}^{(3,0,0)}_{-}$ has basis $\{x_1^3, x_2^3, x_3^3\}$, as does $\mathbb{P}^{(3,0,0)}$, but the latter is not closed under the action of $H_n(q)$.

Then in $\mathbb{P}^{\lambda}_{-}$ all but the first and last terms in the bottom line of equation (3) can be ignored and we have

$$g_i z^{(\ldots,a_i,a_{i+1},\ldots)} = \begin{cases} -q z^{(\ldots,a_i,a_{i+1},\ldots)} & a_i = a_{i+1} \\ (q^{-1} - q) z^{(\ldots,a_i,a_{i+1},\ldots)} - q z^{(\ldots,a_{i+1},a_i,\ldots)} & a_i > a_{i+1} \\ -q^{-1} z^{(\ldots,a_{i+1},a_i,\ldots)} & a_i < a_{i+1}. \end{cases} \tag{8}$$

Note that within $\mathbb{P}^{\lambda}_{-}$ the space induced by the action of the generators on a single z^a is spanned by monomials of the form $z^{p(a)}$, where $p(a)$ is a permutation of the elements of a, (i.e. all monomials of the same shape) and this space is the whole of $\mathbb{P}^{\lambda}_{-}$. Thus

$$\dim(\mathbb{P}^{\lambda}_{-}) = \frac{n!}{\prod_i ((\rho^{\lambda})_i!)}. \tag{9}$$

Note that we can write the basis states in terms of the action of the generators on a 'lowest' state z^e of shape λ defined by $e_i \leqslant e_{i+1}$. Recall the partial order on $\mathbb{N}^n$ defined [15] by $a \succeq b$ if

$$\sum_{i=1}^{j} a_i \geqslant \sum_{i=1}^{j} b_i \qquad \text{for all } j.$$

Then $-q g_i z^b = z^{((i\ i+1)b)}$ for $((i\ i+1)b) \succ b$.

Proposition 3. As $H_n(q)$ modules $\mathbb{P}^{\lambda}_{-} \cong \mathbb{P}^{\mu}_{-}$ if $\rho^{\lambda} = \rho^{\mu}$.

For example: $\mathbb{P}^{(9,6,4,1)} \cong \mathbb{P}^{(8,7,3,2)}$ since if $n = 4$ then $\rho = (1, 1, 1, 1)$ in both cases; if $n = 6$ $\mathbb{P}^{(9,6,4,1,0,0)} \cong \mathbb{P}^{(8,7,3,2,0,0)}$ similarly since $\rho = (2, 1, 1, 1, 1)$ in both cases, and so on.

Proof. Consider the specialization $q = 1$. Then in their monomial bases $\mathbb{P}^{\lambda}_{-}$ and $\mathbb{P}^{\mu}_{-}$ induce representations identical up to permutation of basis states. This is also true generally if

$$\{i : \lambda_i > \lambda_{i+1}\} = \{i : \mu_i > \mu_{i+1}\}$$

(e.g. for $(2, 2, 0)$ and $(1, 1, 0)$, but not for $(2, 2, 0)$ and $(1, 0, 0)$). In other cases the representations are not identical, but the proposition follows by continuity with the $q = 1$ specialization as follows.

Recall that the specialization of $H_n(q)$ to $q = 1$ is the group algebra of the symmetric group—a semi-simple algebra with dimension $n!$. Any element in the radical of $H_n(q)$ must either vanish or again be in the radical of any specialization. Since $H_n(q)$ is also of dimension $n!$ [18] then $\mathrm{rad}(H_n(q))$ is empty. Now suppose $\{R_i\}$ is a set of representatives of equivalence classes of irreducible representations of $H_n(q)$—so $\sum_i |R_i|^2 = n!$. If any such irreducible specializes to a reducible representation then some dimensions are lost, therefore specialization is an isomorphism of classes of irreducibles. Thus the irreducible content at $q = 1$ determines the $H_n(q)$ irreducible content. $\square$

Note that shapes of different degree may have the same profile (e.g. for $n = 3$ $(7, 1, 0)$ and $(3, 2, 1)$ are both $\rho^\lambda = (1, 1, 1)$). There is a unique representative shape in each isomorphism class of minimum degree given by

$$\lambda = (d-1, d-1, \ldots, d-1, d-2, d-2, \ldots,\\ d-2, d-3, \ldots, d-3, \ldots, 1, 1, \ldots, 1, 0, 0, \ldots, 0) \tag{10}$$

where d is the depth of ρ^λ and

$$\mathrm{card}\{i : \lambda_i = k-1\} = (\rho^\lambda)_k$$

(so $|\lambda| = \sum_{i=1}^{d} (i-1).\rho_i^\lambda$).

We will also define a *standard* representative shape ν for each class by

$$\mathrm{card}\{i : \nu_i = k\} = (\rho^\nu)_{d-k}. \tag{11}$$

We will see in the next section that the action in equation (8) (for λ a standard representative shape) is identical to the action of the generators on a ρ^λ q-permutation module [8], which is in turn a block of the $sl(n)$ vertex model representation of $H_n(q)$ ([16, 17] and see review below). The generic irreducible content of such a block is known [18], being given by the Littlewood–Richardson rules [15]. Serendipitously we will be able to rederive this result in a relatively simple way.

4. Polynomial bases for Specht modules

Recall the (unnormalized) q-symmetrizer and q-antisymmetrizer in $H_n(q)$:

$$Y_n^s = \sum_{w \in S_n} (q)^{-l(w)} G(w) \tag{12}$$

$$Y_n^a = \sum_{w \in S_n} (-q)^{+l(w)} G(w) \tag{13}$$

where $G(S_n)$ is the basis of $H_n(q)$ obtained by writing each $w \in S_n$ as a word of minimal length ($l(w)$) in permutations $(i\ i+1) \in S_n$ and then replacing $(i\ i+1) \mapsto g_i$.

These have the properties

$$\left(Y_n{}^{s}_{a} \right)^2 = [n]_\pm! Y_n{}^{s}_{a} \tag{14}$$

(where $[n]_\pm! = \prod_{r=1}^{n} (1 - q^{\mp 2r})/(1 - q^{\mp 2})$) and

$$g_i Y_n{}^{s}_{a} = \pm q^{\mp 1} Y_n{}^{s}_{a}. \tag{15}$$

Proof of equation (15). First consider some g_i acting on an arbitrary summand of Y_n^s. There are two cases to consider—either $g_i G(w)$ is of minimum length (and is given by $G((i\ i+1)w)$); or the relation $g_i^2 = (q^{-1} - q) g_i + 1$ can be used to shorten it:

$$g_i G(w) = \begin{cases} (q^{-1} - q) G(w) + G((i\ i+1)w) & l((i\ i+1)w) = l(w) - 1 \\ G((i\ i+1)w) & l((i\ i+1)w) = l(w) + 1 \end{cases}$$

so overall the coefficient of $G(w)$ on the left-hand side of $g_i Y_n^s$ is again one of two cases—either the coefficient comes from shortening some w' (hence $q^{-l(w)-1}$) or from lengthening w' plus leaving alone w:

$$\begin{array}{ll} q^{-l(w)+1} + q^{-l(w)}(q^{-1} - q) & l((i\ i+1)w) = l(w) - 1 \\ q^{-l(w)-1} & l((i\ i+1)w) = l(w) + 1 \end{array}$$

and similarly for the antisymmetric case. □

There is a duality between the roles of the two types of operator (under $q \leftrightarrow -q^{-1}$), and in fact the normalization we have chosen causes them to exchange their normal roles in the $q = 1$ limit.

Proposition 4. For $(ij) \in S_n$ and f a function of n variables such that $f = (ij)f$, then

$$Y_n^s f = 0.$$

Proof. It is sufficient to prove that $(i\ i+k)f = f$ implies $Y_n^s f = 0$.

We proceed by induction on k. First suppose $f = f^{(i)}$, then

$$(q^{-1} + q)Y^s f \overset{\text{eqn (15)}}{=} Y^s(g_i + q)f \overset{\text{eqn (2)}}{=} 0.$$

This establishes the case $k = 1$ (strictly speaking only for $q \neq \pm i$, although these cases can also be dealt with).

Now assume that $f = (i\ i+m)f$ implies $Y^s f = 0$ for all i and all $m = 1, 2, \ldots, k-1$. Note that if $f = (i\ i+k)f$ then

$$f = ((i\ i+1)(i+1\ i+k)(i\ i+1))f = (i\ i+1)\,((i+1\ i+k)(i\ i+1))\,f$$

and hence

$$(i\ i+1)f = (i+1\ i+k)((i\ i+1)f)$$

so then $Y_n^s((i\ i+1)f) = 0$ by assumption. But similarly $Y_n^s(f + (i\ i+1)f) = 0$, so finally $Y^s f = 0$. □

Writing $f = z^a$ then if two exponents a_i, a_j are the same $(ij)f = f$, so

Proposition 5. For $a \in \mathbb{N}^n$

$$Y_n^s z^a = 0 \qquad \text{unless } \rho^a = (1^n).$$

That is, $Y_n^s\mathbb{P}$ contains no monomial in which two exponents a_i, a_j (say) are equal.

In particular, $Y_n^s\mathbb{P}^i = 0$ unless $i \geqslant n(n-1)/2$. Now consider $Y_n^s\mathbb{P}^{n(n-1)/2}$. First $Y_2^a = 1 - qg_1$ so

$$Y_2^a z_1 = q^2(z_1 + z_2)$$

and so

$$g_1(z_1 + z_2) = -q(z_1 + z_2) \tag{16}$$

and

$$g_1(z_1 - q^2 z_2) = q^{-1}(z_1 - q^2 z_2). \tag{17}$$

This leads us to form:

Definition (q-Vandermondians). The q-Vandermondians [15] are

$$V_n^{\pm}(q) = \prod_{1\leqslant i<j\leqslant n} (z_i \mp q^{1\pm 1} z_j) = V_{n-1}^{\pm}(q)\left(\prod_{i=1}^{n-1}(z_i \mp q^{1\pm 1} z_n)\right).$$

These definitions with equations (16) and (17) give

$$g_i V_n^{\pm}(q) = \pm q^{\mp 1} V_n^{\pm}(q). \tag{18}$$

Note that $V_n^{\pm}(q) \in \mathbb{P}_+^{(n-1,n-2,\dots,0)} \subseteq \mathbb{P}^{n(n-1)/2}$. For example, the first term in the monomial expansion of $V_n^{\pm}(q)$ is $z_1^{n-1} z_2^{n-2} \dots z_{n-1}$ (i.e. profile (1^n)). Note also that as left $H_n(q)$ modules

$$\mathbb{Z}_q Y_n^{\,\overset{s}{a}} \cong \mathbb{Z}_q V_n^{\pm}(q). \tag{19}$$

Whereupon we have:

Corollary 5.1. For $k \geqslant n$ let $\lambda = (k-1, k-2, \dots, k-n)$, then as a left $H_n(q)$ module

$$Y_n^s \mathbb{P}_+^{\lambda} \cong \mathbb{Z}_q V_n^{+}(q)\left(\prod_{i=1}^{n} z_i^{k-n}\right). \tag{20}$$

Proof. For some z^a of shape λ let us write $X = Y_n^s z^a \in \mathbb{P}_+^{\lambda}$. By proposition 4

$$Y_n^s(1 + (ij))z^a = 0$$

so for b any (even/odd) permutation of a

$$Y_n^s z^b = \pm X.$$

By proposition 5 $Y_n^s z^c = 0$ for $c^{\square} \lhd \lambda$, since shapes of depth $\leqslant n$ which are lower than λ in the total order (of partitions of $n(k-n) + n(n-1)/2$) must have at least two exponents equal. Thus

$$Y_n^s \mathbb{P}_+^{\lambda} = \mathbb{Z}_q X.$$

But $V_n^{+}(q)(\prod_{i=1}^{n} z_i^{k-n}) \in \mathbb{P}_+^{\lambda}$ (consider $k = n$) and, comparing the coefficients of maximum degree in the first n variables in $Y_n^s V_k^{+}(q) = [n]_+! V_k^{+}(q)$ we get

$$Y_n^s V_n^{+}(q)\left(\prod_{i=1}^{n} z_i^{k-n}\right) = [n]_+! V_n^{+}(q)\left(\prod_{i=1}^{n} z_i^{k-n}\right).$$

□

Further:

Definition 4 (Diagrams for Vandermondians). More explicitly $V_n^{\pm}(q) = V_n^{\pm}(q, z_1, z_2, \dots, z_n)$ and then the tableau Vandermondians are

$$\begin{array}{|c|}\hline m{+}1 \\ \hline m{+}2 \\ \hline \vdots \\ \hline m{+}k \\ \hline \end{array}^{\,+} = V_k^{+}(q, z_{m+1}, z_{m+2}, \dots, z_{m+k})$$

$$\begin{array}{|c|c|c|c|c|}\hline m{+}1 & m{+}2 & m{+}3 & \quad\dots\quad & m{+}k \\ \hline \end{array}^{\,-} = V_k^{-}(q, z_{m+1}, z_{m+2}, \dots, z_{m+k})$$

and similarly

$$\begin{array}{|c|c|c|c|c|}\hline m & a & c & .. & e \\ \hline \vdots & \vdots & \vdots & & f \\ \hline m{+}k & b & d & & \\ \hline \end{array}^{\,+} = \begin{array}{|c|}\hline m \\ \hline \vdots \\ \hline m{+}k \\ \hline \end{array}^{\,+} \times \begin{array}{|c|c|c|c|}\hline a & c & .. & e \\ \hline \vdots & \vdots & & f \\ \hline b & d & & \\ \hline \end{array}^{\,+}$$

and

$$\begin{array}{|c|c|c|c|c|}\hline a & .. & .. & & b \\ \hline c & .. & & d & \\ \hline m & .. & m{+}k & & \\ \hline \end{array}^{\,-} = \begin{array}{|c|c|c|c|c|}\hline a & .. & .. & & b \\ \hline c & .. & & d & \\ \hline \end{array}^{\,-} \times \begin{array}{|c|c|c|}\hline m & \ldots & m{+}k \\ \hline \end{array}^{\,-}$$

and so on.

Definition 5 (Tableau symmetrizers). Similarly, diagrams as above but with superscripts $\pm$ replaced by a, s denote the elements of $H_n(q)$ obtained by replacing the tableau Vandermondians with the corresponding 'translated' (anti)symmetrizers, i.e. $V_n^{\pm}(q) \to Y_n{}^{\substack{s\\a}}$.

For example,

$$\begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array}^{\,a} = 1 - qg_3.$$

Let us review the construction of q-permutation blocks and Specht modules [9]. In order to do this we need to define some more special elements of $H_n(q)$. Let $\mu \vdash n$ have depth $\mu_1' = d$, then with Λ_μ^a the product of symmetrisers defined by

$$\Lambda_\mu^a = \begin{array}{|c|c|c|c|c|c|c|}\hline n{-}\mu_1{+}1 & n{-}\mu_1{+}2 & \cdots & & & & n \\ \hline \vdots & & & & & & \\ \hline \mu_d{+}1 & \mu_d{+}2 & \cdots & & \mu_d{+}\mu_{d-1} & & \\ \hline 1 & 2 & \cdots & \mu_d & & & \\ \hline \end{array}^{\,a} \tag{21}$$

we have:

Definition 6 (q-permutation block [9, 18]). The left module $P_\mu = H_n(q)\Lambda_\mu^a$ is the q-permutation block associated with μ.

Proposition 6. As left $H_n(q)$ modules, P_μ is isomorphic to $\mathbb{P}_-^\lambda$ in case the profile of λ is $\rho^\lambda = \mu$.

Proof. The induced representations are identical in a canonical basis [18] if λ is standard. In particular compare equation (8) (for the lowest state) with equation (21).

Let Λ_μ^s be the product of symmetrisers defined by

$$\Lambda_\mu^s = \begin{array}{|c|c|c|c|c|c|c|}\hline 1 & \mu_1'{+}1 & \cdots & & & & n \\ \hline 2 & \mu_1'{+}2 & \cdots & & & & \\ \hline \vdots & \vdots & & & & & \\ \hline \mu_1' & \mu_1'{+}\mu_2' & \cdots & & & & \\ \hline \end{array}^{\,s} \tag{22}$$

(in the illustration $\mu_1' = \mu_2'$, of course this need not be true in general) and let $\Lambda_\mu^\mp$ be the tableau Vandermondians obtained from the tableaux in equations (21) and (22) respectively.

Definition 7 (q-Specht module [8, 9, 18]). The left module $H_n(q)\Lambda^s_\mu P_\mu$ is the q-Specht module associated with μ.

Each such module is a simple $H_n(q)$ module as we will see, and well defined in any specialization. Now by proposition 5:

Proposition 7. For $\mu \vdash n$

$$\Lambda^s_\mu z^a = 0 \tag{23}$$

unless $(\rho^a)' \trianglerighteq \mu'$. Further, if a is of minimal degree for profile ρ^a (equation (10)) then

$$\Lambda^s_{\rho^a} z^a \propto \Lambda^+_{\rho^a} \tag{24}$$

that is

$$\Lambda^s_{\rho^a}\mathbb{P}^a_+ = \mathbb{Z}_q \Lambda^+_{\rho^a} \tag{25}$$

and if the shape of a is standard (equation (11)) then

$$\Lambda^s_{\rho^a} z^a \propto \Lambda^+_{\rho^a}\left(\prod_{i=1}^{a_1}\left(\prod_{j=1}^{a'_i} z_j^{i-1}\right)\right) \tag{26}$$

that is

$$\Lambda^s_{\rho^a}\mathbb{P}^a_+ = \mathbb{Z}_q \Lambda^+_{\rho^a}\left(\prod_{i=1}^{a_1}\left(\prod_{j=1}^{a'_i} z_j^{i-1}\right)\right). \tag{27}$$

Note that equation (23) uses an ordering on *profiles* (cf the ordering on shapes in proposition 1).

Proof. Explicitly

$$\Lambda^s_\mu z^a = (Y^s_{\mu'_1} z^{(a_1,a_2,\dots,a_{\mu'_1},0,0,\dots,0)}) \cdot \left(\begin{array}{|c|} \hline \mu'_1+1 \\ \hline \vdots \\ \hline \mu'_1+\mu'_2 \\ \hline \end{array}^{\,s} z^{(0,0,\dots,0,a_{\mu'_1+1},a_{\mu'_1+2},\dots,a_{\mu'_1+\mu'_2},0,0,\dots,0)} \right) \cdot (\dots).$$

Let the depth of ρ^a be $(\rho^a)'_1 = d$. In order for the first factor on the right-hand side to be non-vanishing the exponents must be some permutation of μ'_1 distinct numbers chosen from the d distinct exponents. If $d < \mu'_1$ this is not possible. If $d > \mu'_1$ then $(\rho^a)' \triangleright \mu'$ and there is nothing to prove. Note that if the exponents are a permutation of $\{0, 1, 2, \dots, \mu'_1 - 1\}$ then the factor is proportional to $V^+_{\mu'_1}$ by corollary 5.1.

It remains to consider the case $d = \mu'_1$. Suppose that the first factor is in fact non-vanishing. In order for the second factor to be non-vanishing the exponents there must be some permutation of μ'_2 distinct numbers. Since we have already factored out μ'_1 distinct exponents we have an effective profile obtained by deleting the first column (in the diagram sense) from ρ^a. Thus the distinct exponents must be chosen from $(\rho^a)'_2$ possibilities. If $(\rho^a)'_2 < \mu'_2$ this is not possible. If $(\rho^a)'_2 > \mu'_2$ then $(\rho^a)' \triangleright \mu'$ and there is nothing to prove. If the exponents are a permutation of $\{0, 1, 2, \dots, \mu'_2 - 1\}$—the minimal degree case—then the factor is proportional to $V^+_{\mu'_2}$ (appropriately translated) by corollary 5.1. If the exponents are a permutation of $\{\mu'_1 - \mu'_2, \dots, \mu'_1 - 2, \mu'_1 - 1\}$—the standard case—then the factor is proportional to $V^+_{\mu'_2}(\prod_{j=1}^{\mu'_2} z_j)$ (translated by $z_i \to z_{i+\mu'_1}$) by corollary 5.1.

In general if $(\rho^a)'_2 = \mu'_2$ we must consider the third factor similarly, and so on. Iterating, we obtain equations (23), (24) and (26).

Finally, consider the action of Λ^s_μ on a monomial z^b (say) in $\mathbb{P}^a_+$, of shape lower than $a^\square$ in the total order (in the case of a of minimal degree for its profile). Then the multiplicity of some exponent is lower in b than in a, but since at minimal degree every exponent between this one and zero is represented, the multiplicity of some lower exponent is necessarily higher in b. Correspondingly at some factor there will not be enough distinct exponents in b, giving $\Lambda^s_\mu z^b = 0$. Equation (27) follows similarly. □

It is proved in [18] that $\Lambda^s_\mu P_\mu$ is one-dimensional for any $\mu \vdash n$. In general if $X, Y \in H_n(q)$ and $XH_n(q)Y$ is one-dimensional then $H_n(q)XH_n(q)Y$ is an indecomposable left module [19], hence the Specht module $H_n\Lambda^s_\mu P_\mu$ is a generic irreducible left module (distinct for each μ, with $\mathcal{P}_n$ giving a complete set of irreducibles up to equivalence). It then follows from proposition 6 that for each λ the space $\Lambda^s_{\rho^\lambda}\mathbb{P}^\lambda_-$ is one-dimensional (and this is confirmed explicitly in the mimimum degree and standard cases by proposition 7, since $\mathbb{P}^\lambda_-$ is a quotient of $\mathbb{P}^\lambda_+$)†. Thus $\Lambda^s_{\rho^\lambda}\mathbb{P}^\lambda_-$ generates a polynomial basis for the corresponding left Specht module—as indeed does $\Lambda^s_{\rho^\lambda}\mathbb{P}^\lambda_+$ in minimum and standard cases. By equation (25) the polynomial $\Lambda^+_{\rho^\lambda}$ in particular generates this basis. For example

$$\Lambda^+_{(2^m)} = \left(\prod_{1\leqslant i<j\leqslant m}(z_i - q^2 z_j)\right)\prod_{m+1\leqslant i<j\leqslant 2m}(z_i - q^2 z_j). \tag{28}$$

Similarly

$$\Lambda^+_{(2^m,1)} = \left(\prod_{1\leqslant i<j\leqslant m+1}(z_i - q^2 z_j)\right)\prod_{m+2\leqslant i<j\leqslant 2m+1}(z_i - q^2 z_j). \tag{29}$$

An isomorphic representation is induced using $\Lambda^+_{(2^m,1)}\prod_{i=m+2}^{2m+1} z_i$ (from equation (27)), reproducing (with equation (28)) the two examples found by Davies *et al* [1]. Equivalently, note for example that $a = (3,3,2,2,1,1,0)$ and $(3,2,2,1,1,0,0)$ have the same profile (i.e. $(2^3,1)$), and that in both cases there are no shapes lower in the total order with profiles higher (or equal) in the total order of their conjugates.

More generally, for example

$$\Lambda^+_{(3^m)} = \left(\prod_{1\leqslant i<j\leqslant m}(z_i - q^2 z_j)\right)\left(\prod_{m+1\leqslant i<j\leqslant 2m}(z_i - q^2 z_j)\right)\prod_{2m+1\leqslant i<j\leqslant 3m}(z_i - q^2 z_j) \tag{30}$$

and so on.

A complete basis is generated from $\Lambda^+_{\rho^\lambda}$ by acting with $\{g_i\}$ as in the following example. We consider $\rho^\lambda = (2^3, 1)$ and write $(ij\ldots k)$ for $g_i g_j \ldots g_k \Lambda^+_{\rho^\lambda}$ Then a basis is

$$\Lambda^+_\lambda = \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 6 \\ \hline 3 & 7 \\ \hline 4 & \\ \cline{1-1}\end{array}^{\,+} \qquad (4) = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 6 \\ \hline 3 & 7 \\ \hline 5 & \\ \cline{1-1}\end{array}^{\,+} \qquad (34) = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 6 \\ \hline 4 & 7 \\ \hline 5 & \\ \cline{1-1}\end{array}^{\,+}$$

† A straightforward generalization of the proof of proposition 7 establishes that the Littlewood–Richardson rules for the Specht module content of permutation blocks [9, 15] continue to apply on generalizing from $q = 1$ to q indeterminate.

$$(54) = \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&7\\\hline \multicolumn{1}{|c|}{6}&\\\cline{1-1}\end{array}^{+} \qquad (654) = \begin{array}{|c|c|}\hline 1&4\\\hline 2&5\\\hline 3&6\\\hline \multicolumn{1}{|c|}{7}&\\\cline{1-1}\end{array}^{+} \qquad (354) = \begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4&7\\\hline \multicolumn{1}{|c|}{6}&\\\cline{1-1}\end{array}^{+} \qquad (234) = \begin{array}{|c|c|}\hline 1&2\\\hline 3&6\\\hline 4&7\\\hline \multicolumn{1}{|c|}{5}&\\\cline{1-1}\end{array}^{+}$$

$$(4354) = \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5&7\\\hline \multicolumn{1}{|c|}{6}&\\\cline{1-1}\end{array}^{+} \qquad (2354) = \begin{array}{|c|c|}\hline 1&2\\\hline 3&5\\\hline 4&7\\\hline \multicolumn{1}{|c|}{6}&\\\cline{1-1}\end{array}^{+} \qquad (6534) = \begin{array}{|c|c|}\hline 1&3\\\hline 2&5\\\hline 4&6\\\hline \multicolumn{1}{|c|}{7}&\\\cline{1-1}\end{array}^{+}$$

$$(24354) = \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&7\\\hline \multicolumn{1}{|c|}{6}&\\\cline{1-1}\end{array}^{+} \qquad (64354) = \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline 5&6\\\hline \multicolumn{1}{|c|}{7}&\\\cline{1-1}\end{array}^{+} \qquad (26354) = \begin{array}{|c|c|}\hline 1&2\\\hline 3&6\\\hline 4&7\\\hline \multicolumn{1}{|c|}{5}&\\\cline{1-1}\end{array}^{+}$$

$$(246354) = \begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline 5&6\\\hline \multicolumn{1}{|c|}{7}&\\\cline{1-1}\end{array}^{+}$$

Finally, the algebra $H_n(q)$ is isomorphic to its opposite (see the defining relations), so an action from the right may be defined similarly to equation (2). Note that, in particular, $\mathbb{P}$ can be regarded as a left or right $H_n(q)$ module, but it is not a bimodule. For example, in the S_n case

$$(12)(z_1(23)) = (12)z_1 = z_2 \qquad \text{but} \qquad ((12)z_1)(23) = z_2(23) = z_3.$$

We have determined the structure of the ring of n-variable polynomials as a Hecke algebra module by identifying various submodules with Hecke modules of known structure. Our new realization of these (q-permutation) modules has enabled us to rederive their structure in a relatively simple way, and in particular to construct polynomial bases for all irreducible (q-Specht) modules.

Acknowledgments

I would like to thank T Miwa, M Jimbo, A Nakayashiki and B Westbury for stimulating conversations. I would like to thank SERC and the Nuffield Foundation for financial support. Thanks are also due to the organizers of the Isaac Newton Institute programme for autumn 1992.

References

[1] Davies B, Foda O, Jimbo M, Miwa T and Nakayashiki A 1993 *Commun. Math. Phys.* **151** 89
[2] Tsuchiya A and Kanie Y 1989 *Advanced Studies in Pure Mathematics* vol 16 (Tokyo: Kinokuniya)
Cherednik I 1992 *Int. J. Mod. Phys.* A **7** Suppl 1A 109 and references therein
[3] Lusztig G 1985 *Proc. Am. Math. Soc.* **94** 337
Lascoux A and Schutzenberger M-P 1982 *C. R. Acad. Sci. Paris* **294** 447; 1987 *Funct. Anal. Appl.* **21** 77
[4] Date E, Jimbo M, Kuniba A, Miwa T and Okado M 1989 *Lett. Math. Phys.* **17** 69
Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523
Deguchi T, Wadati M and Akutsu Y 1988 *J. Phys. Soc. Japan* **57** 2921
[5] Hoefsmit P N 1974 *PhD Thesis* University of British Columbia
[6] Andrews G E, Baxter R J and Forrester P J 1984 *J Stat. Phys.* **35** 193
Huse D A 1984 *Phys. Rev.* B **30** 3908

[7] Gyoja A 1986 *Osaka J. Math.* **23** 841
Wenzl H 1988 *Invent. Math.* **92** 349
[8] James G and Murphy G E 1979 *J. Algebra* **59** 222
Dipper R and James G 1987 *Proc. London Math. Soc.* **54** 57 and references therein
[9] James G D and Kerber A 1981 *The Representation Theory of the Symmetric Group* (Reading, MA: Addison-Wesley)
Dipper R and James G 1986 *Proc. London Math. Soc.* **52** 20
[10] Smirnov F A 1992 *Int. J. Mod. Phys.* A **7** Suppl 1B 813, 839
Bernard D and LeClair A 1991 *Commun. Math. Phys.* **142** 99
[11] Frenkel I B and Reshetikhin N Yu 1992 *Commun. Math. Phys.* **146** 1
[12] Calogero F 1971 *J. Math. Phys.* **12** 419
Sutherland B 1971 *J. Math. Phys.* **12** 246, 251; 1971 *Phys. Rev.* A **4** 2019; 1972 *Phys. Rev.* A **5** 1372; 1992 *Phys. Rev.* B **45** 907
Moser J 1975 *Adv. Math.* **16** 197
Sogo K 1993 Excited states of CSM model—classification by Young diagrams *Preprint* Tokyo University
[13] Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523 and references therein
[14] Babelon O, de Vega H J and Viallet C M 1983 *Nucl. Phys.* B **220** 13
[15] MacDonald I G 1979 *Symmetric Functions and Hall Polynomials* (Oxford: Clarendon)
[16] Temperley H N V and Lieb E H 1971 *Proc. R. Soc.* A **322** 251
[17] Date E, Jimbo M, Miwa T and Okado M 1987 *Nucl. Phys.* B **290** 231
[18] Martin P P 1992 *Int. J. Mod. Phys.* A **7** Suppl 645
Deguchi T and Martin P P 1992 *Int. J. Mod. Phys.* A **7** Suppl 165
[19] Martin P P 1991 *Potts Models and Related Problems in Statistical Mechanics* (Singapore: World Scientific)